ON SYMPLECTIC PRIMITIVE MODULES AND MONOMIAL GROUPS

BY

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INTRODUCTION

In [7], D. T. Price determines most of the structure of the so-called minimal non-M-groups. By definition a minimal non-M-group is a finite non-monomial group all of whose proper sections are monomial. A finite group T is said to be monomial, when all irreducible representations of T over the complex field are induced by linear representations of subgroups of T.

The theorem proved by Price is the following:

THEOREM 1. *Let G be a finite solvable minimal non-M-group. Then the following properties hold*:

1. *$G=FA$, where F is an extra-special normal p-subgroup of G of exponent p, p prime (when $p=2$, then F is not dihedral) and the subgroup A of G acts irreducibly on $F/Z(F)$ and trivially on $Z(F)$.*
2. *$O_{p'}(G)=\{1\}$.*
3. *Either A is some p'-group or $p=2$ and $A/O_{2'}(A)$ is a cyclic 2-group.*
4. *If A is of odd order then A has prime order.*

As done in [8], it can easily be shown that the Fitting subgroup $F(G)$ of G is equal to the group $Z(G)F$ and $Z(G)$ is a cyclic p-group. {Indeed, if $p\neq 2$, or if $p=2$ and A is a $2'$-group, then $F(G)=F=FZ(G)$ by properties 2) and 3) of the just mentioned theorem, and when $p=2$ and A is not a $2'$-group, then by property 3) of the theorem it follows that $F(G)/F$ is a cyclic 2-group. Use in the latter case the construction of [8], page 161, and the result follows}.

It is the purpose of this paper to clearify the structure of the group A. More precisely, since it can easily be seen that $F(G)=C_G(F/Z(F))$, we study the structure of $G/F(G)$, with $G/F(G)$ considered as subgroup of Aut $(F/Z(F))$.

We prove the following

THEOREM 2. *Let G be a finite solvable minimal non-M-group. Assume that $G/F(G)$ does not contain a normal extra-special r-subgroup of exponent r for any odd prime r. Then $G/F(G)$ is one of the following groups*:

1. *$G/F(G)$ is cyclic of odd prime order q,*
2. *$G/F(G)$ is quaternion of order 8,*
3. *$G/F(G)$ is dihedral of order $2q$, q odd prime,*
4. *$G/F(G) = \langle x, u | u^4 = 1 = x^q,\ u^{-1}xu = x^{-1}\rangle$, q odd prime,*
5. *$G/F(G)$ is a cyclic group of order 4.*

In the course of the proof of theorem 2 we make the action of $G/F(G)$ on $F/Z(F)$ precise, in particular the prime p dividing $|F|$ plays an important role just as the prime q does, which occurs in this theorem 2. The minimal non-M-groups corresponding to theorem 2 are described in § 3.

Notations and conventions are standard and can be found in [4], [6] and [8]. All groups in this paper are finite; the word "finite" will be omitted from now on.

§ 1. SYMPLECTIC PRIMITIVE MODULES

Let G be a minimal non-M-group which is solvable. As we have seen, there exists a normal extra-special p-subgroup F of G. For any two elements $x, y \in F$ we have $[x, y] = z^i$ with a fixed z such that $\langle z\rangle = Z(F)$. Now $Z(F) \subseteq Z(G)$, whence $i \in \mathbf{F}_p$ depends only on the class $\bar{x}$ of x mod $Z(F)$ and on the class $\bar{y}$ of y mod $Z(F)$. $F/Z(F)$ is elementary abelian, and so $F/Z(F)$ can be viewed as an irreducible $\mathbf{F}_p(G/F(G))$-vector space V. The map $f\colon V \times V \mapsto \mathbf{F}_p^+$, given by $(\bar{x}, \bar{y}) \mapsto i$, with $\bar{x}, \bar{y}$ considered as elements of the vector space V by abuse of language, i is defined above, induces a non-singular symplectic form on V, denoted again by $(\, , \,)$. The group $G/F(G)$ acts on the vector space V leaving the symplectic form $(\, , \,)$ invariant, i.e. when $t \in G/F(G)$ then $(\bar{x}^t, \bar{y}^t) = (\bar{x}, \bar{y})$ for all $\bar{x}, \bar{y} \in V$. This follows from the group action

$$[x^t, y^t] = [t^{-1}xt, t^{-1}yt] = (t^{-1}xt)^{-1}(t^{-1}yt)^{-1}(t^{-1}xt)(t^{-1}yt) =$$
$$= t^{-1}(x^{-1}y^{-1}xy)\,t = x^{-1}y^{-1}xy = [x, y] \text{ for any } t \in G.$$

Therefore we can view $G/F(G)$ as a subgroup of $Sp(2m, p)$ with $|F| = p^{2m+1}$. $\{p$ divides $|F|$ with an odd power; see [6], Th. III. 13.7$\}$.

Define $N = G/F(G)$, $H_1 = HF(G)/F(G)$ with H a proper subgroup of G. Let V_1 be an irreducible $\mathbf{F}_pH_1$-submodule of V. If $H_1 = N$ then $V = V_1$; in that case HF acts irreducibly on $F/Z(F)$, and as any proper subgroup of G is monomial, it follows by arguments of [7] and [8], that $HF = G$. If H_1 is a proper subgroup of N, then $HF(G)$ is monomial and putting $F_1/Z(F)$ the subgroup of $F/Z(F)$ corresponding to V_1, then $F_1 \neq F$ by the same argument as in the last sentence following [7] and [8]. Now HF/F acts irreducibly on $F_1/Z(F)$ and trivially on $Z(F)$. By the structure of G/F, and by the fact that HF is monomial it follows that F_1 cannot be non-abelian. {Indeed, otherwise F_1 would be extra-special and by [7] it would follow that HF is not monomial via the theory of extension of characters}. So F_1 is abelian, that is, the symplectic form (,) on V is trivial on V_1.

Therefore we have the following general situation in which we use the same symbols as above to stress the analogy. We are given the group N; $\mathbf{F}_p$ is the field of p elements, p prime; V is an irreducible $\mathbf{F}_p$ N-module, g is a non-singular symplectic form on V fixed by N. We say that W symplectically induces V if

1. W is a $\mathbf{F}_pJ$-submodule of V for a subgroup J of N,
2. g is non-singular on W,
3. $V \cong W^N$, the induced module,
4. if $x, y \in N$ where $xJ \neq yJ$, then xW and yW are orthogonal.

We say that V is symplectic primitive if V is not symplectically induced by any proper (irreducible) subspace W of V. In our special situation this is indeed the case, but notice that "our" V is just an *example* of a symplectic primitive module.

From Berger's papers [1] and [2], it follows immediately that the next proposition holds:

PROPOSITION. *An irreducible symplectic primitive* $\mathbf{F}_pN$*-module* V *is also a minimal* $\mathbf{F}_p$ N*-module.*

We recall the definition of "minimal KR-module". Suppose R is a group, K is a field, X is an irreducible KR-module. Let g be a non-singular symplectic form on X fixed by R. Then X is called a minimal KR-module if one of the following properties holds:

1. $X|_S$ is homogeneous for all normal subgroups S of R, or
2. for any normal subgroup S of R such that $X|_S \cong X_1 \dotplus \ldots \dotplus X_t$, where $t > 1$ and the X_i are distinct homogeneous components then
   (i) $t=2$;
   (ii) X_i is totally isotropic for $i=1, 2$; and
   (iii) $X_1|_L$ is homogeneous for all normal subgroups L of R contained in the stabilizer of X_1 in R.

Putting $X=V$, $K=\mathbf{F}_p$, $R=N=G/F(G)$, we are back in our special situation.

§ 2. SYMPLECTIC PRIMITIVE MODULES IN CONNECTION WITH THEOREM 2

According to what we have to prove in theorem 2, let $N=G/F(G)$ have no normal extra-special r-subgroup of exponent r for any odd prime r dividing $|N|$. Theorem 2.9 of [7] states that any normal abelian subgroup of N is cyclic. Therefore any characteristic abelian subgroup of $F(N)$ is cyclic. In particular, it follows from a theorem of P. Hall ([6], Th. III. 13.10) that, if s is an odd prime dividing $|F(N)|$, the Sylow s-subgroup P of $F(N)$ equals $C_{s^a} \curlyvee M$ with M an extra-special s-group of exponent s if P is not abelian, otherwise P itself is cyclic.

Assume $P=C_{s^a} \curlyvee M$. Then notice that $C_{s^a}=Z(P)$ is of order s^a and that P is characteristic in $F(N)$ whence normal in N. Let $m \in M$ and let $\alpha \in \mathrm{Aut}(P)$. Then $m^\alpha = nc$ for some $n \in M$ and $c \in C_{s^a}$. Hence $(m^s)^\alpha = (nc)^s = n^s c^s$. As $m \in M$, $n \in M$, it follows that $m^s = n^s = 1$. Thus $c^s = 1$. Then, however, by the amalgamation, $c \in M$. Hence M is characteristic in P, thus normal in N. By assumption such a group M does not fit as normal subgroup of N. It follows that P must be cyclic.

Next, let $2 \mid |F(N)|$. Any characteristic abelian subgroup of the Sylow 2-subgroup P_2 of $F(N)$ is cyclic, as it is also a normal abelian subgroup of N. We can apply P. Hall's theorem again and we obtain: P_2 is cyclic, generalized quaternion, semi-dihedral, dihedral or extra-special, or P_2 equals $U \curlyvee R$, with U extra-special, R equal to a cyclic, or generalized quaternion, or semi-dihedral, or dihedral 2-group. If such a non-abelian subgroup P_2 occurs then by lemma 1.7 of [7], $F(G)$ has odd prime power order p^u. Assume P_2 contains a quaternion subgroup of order 8. Then, if $p \equiv 1 \pmod 4$, we can apply lemmas 2.2 and 2.3 of [7], from which it follows that immediately $N = Q$, the quaternion group of order 8, thereby settling the case 2) of theorem 2. For the action of Q on $F(G)$, see § 3. If $p \equiv -1 \pmod 4$, then we can apply lemma 2.3 and lemma 2.2.b of [7], so $N = C_4$ as Q contains a cyclic subgroup C_4. This is, however, a contradiction, and this case does not occur *here*! By Hall's theorem, there remains P_2 dihedral or cyclic. We split up:

α) Let P_2 be a dihedral 2-group. Then $\mathrm{Aut}(P_2)$ is a 2-group. Above we have found that $F(N)$ is a direct product of P_2 with the cyclic Sylow subgroups for the odd primes dividing $|F(N)|$. Now $N/C_N(F(N))$ can be viewed as a subgroup of $\mathrm{Aut}(F(N))$ and $\mathrm{Aut}(F(N))$ is a direct product of the automorphism groups of the Sylow subgroups of $F(N)$. By the structure of the Sylow subgroups of $F(N)$, it follows that $\mathrm{Aut}(F(N))$ is nilpotent. Since $F(N) \supseteq C_N(F(N))$, $N/F(N)$ is nilpotent. Consider $F(N)/O_{2'}(F(N))$ and let $X/O_{2'}(F(N))$ be the unique maximal cyclic subgroup of the dihedral 2-group $F(N)/O_{2'}(F(N))$. Then it follows easily that $X \lhd N$. Further $F(N)/X$ is a normal subgroup of N/X of order 2. $N/F(N)$ is nilpotent. Assume that N/X is not nilpotent. Then by Th. VI. 7.15 of [6], N/X splits over $F(N)/X$ and $F(N)/X$ is a direct factor of N/X. So N/X would be nilpotent, and this produces a contradiction. Hence N/X is always nilpotent! Since X is the direct product of its (cyclic) Sylow subgroups, X is cyclic. We keep this in mind and we proceed with the second case.

β) Let P_2 be cyclic. By what we have seen before, the Sylow t-subgroups of $F(N)$ are here cyclic for all primes t dividing $|F(N)|$. Therefore $F(N)$ itself is cyclic, and $N/F(N) = N/C_N(F(N))$ is a subgroup of the group $\mathrm{Aut}(F(N))$ which is abelian in this case.

In either case α), β) N contains a non-trivial cyclic self-centralizing normal subgroup T such that N/T is nilpotent. In case α) $T = X$, in case β) $T = F(N)$. For the sake of completeness we mention that this statement is also true, as we have seen, if $2 \nmid |F(N)|$.

Now we have arrived precisely at Hypothesis (3.1) of Berger's paper [1] on minimal modules. Namely, Berger's hypothesis (3.1) is:

HYPOTHESIS. *Let $K=\mathbf{F}_p$ for a prime p, and assume the following four facts are given*:

(1) *V is an irreducible $2m$-dimensional $\mathbf{F}_p$-vector space.*

(2) *$g\colon V\times V\to\mathbf{F}_p$ is a non-singular symplectic form upon V.*

(3) *$N=BL$ is a group with normal cyclic subgroup B and nilpotent complement L, where $B\cap L=\{1\}$.*

(4) *V is a faithful minimal $\mathbf{F}_pN$-module where N fixes the form g, i.e. $g(xu,v)=g(u,x^{-1}v)$ for all $x\in N$, $u,v\in V$.*

Notice that by our results obtained above, it is indeed true that $N^{\mathfrak{N}}$, the nilpotent residual of N, is cyclic and so N is a semi-direct product of $N^{\mathfrak{N}}$ with the nilpotent group M with $M\subseteq N$ and $M\cong N/N^{\mathfrak{N}}$; see Th. VI. 7.15 of [6]. So (3) of the hypothesis is fulfilled.

We split up again: γ) $N^{\mathfrak{N}}=\{1\}$; δ) $N^{\mathfrak{N}}\neq\{1\}$.

γ) Let $N^{\mathfrak{N}}=\{1\}$, i.e. N is nilpotent. We use the subgroup T defined above. "Our" V is a symplectic primitive $\mathbf{F}_pN$-module whence also a minimal $\mathbf{F}_pN$-module. Following Berger's paper [1] we see that the following two cases can arise: γ,1) $V|_T$ is homogeneous; γ,2) $V|_T=W_1\dotplus W_2$ where W_i is a homogeneous totally isotropic $\mathbf{F}_pT$-submodule of V and where W_2 is the dual of W_1. We treat these cases seperately:

Re γ,1) Let $V|_T$ be homogeneous. Then N can be cyclic or N is a generalized quaternion 2-group. Assume N is cyclic, and let $|N|$ be divisible by an odd prime q. By [1], N is a subgroup of the cyclic group C of order p^m+1 with p and m as in the Hypothesis. (So $|F|=p^{2m+1}$). Let D be a subgroup of N of prime order q. V is a completely reducible $\mathbf{F}_pD$-module. Let W be an irreducible $\mathbf{F}_pD$-submodule of V. If $\dim_{\mathbf{F}_p}(W)=\alpha$, then α is the smallest natural number such that $p^\alpha\equiv 1 \pmod q$. Hence q divides $(p^m+1,\,p^\alpha-1)$. If α is even, then by Price's theorem 2.6 of [7], it follows that $N=D$, and this settles the case 1) of theorem 2. Hence assume α is odd. Now $N=T$ and $V|_L$ is homogeneous for any (normal) subgroup L of N, in particular for D. Hence $V|_D$ is the direct sum of isomorphic copies of W, so we have $\alpha|2m$. Since α is odd by assumption, $\alpha|m$. Hence q divides p^m+1 and p^m-1, so we find the contradiction $q=2$. Next, let N be cyclic and assume $|N|=2^\varrho$. We have $2^\varrho|p^m+1$. Here p is odd. Is $p\equiv 1 \pmod 4$ then $\varrho=1$. However, $G/F(G)$ being then of order 2, cannot act irreducibly on the vector space $F/Z(F)$. So we must have $p\equiv 3 \pmod 4$. By the lemmas 2.2 and 2.3 of [7], $N=C_4$ as soon as N contains a cyclic subgroup of order 4. Again $N\neq C_2$, so we have settled the case 5) of the theorem 2.

Next, let N be generalized quaternion. Then $p^m\equiv -1 \pmod 4$, according to point (3.11) of [1]. But then $p\equiv 3 \pmod 4$ and N contains a cyclic subgroup of order 4. Again it would follow that $N=C_4$, but now it yields a contradiction.

Re γ,2) Let $V|_T = W_1 \dotplus W_2$. By point (3.19) of [1] there are in principle two possibilities for N, namely

γ,2,1) N is generalized quaternion with $p^m \equiv 1 \pmod 4$. Since V is symplectic primitive, $m=1$ by point (4.5) of [1]. By lemma 2.2 of [7], N is quaternion of order 8 since N contains such a subgroup. This settles again the case 2) of theorem 2. Note that $|F| = p^3$. See further § 3.

γ,2,2) N is a subgroup of $A \times B$ where A is some generalized quaternion, dihedral or semi-dihedral 2-group, B is cyclic of odd order and N is not a 2-group. See (4.6) of [1]. N contains a non-trivial normal $2'$-subgroup M, which is cyclic. Let ω be some primitive $|M|^{\text{th}}$-root of unity modulo p. By [1], point (2.4), $|M|$ divides $(p^n+1)/2$, where $m=2n$ and $\dim_{\mathbf{F}_p}(V) = 2m$. As the stabilizer $T(W_1)$ of W_1 in N is of index 2 in N, we see that M stabilizes W_1. Let U_1 be an irreducible $\mathbf{F}_pM$-submodule of W_1. As $W_1|_M$ is homogeneous, $\dim_{\mathbf{F}_p}(U_1)|2n$. Let $\dim_{\mathbf{F}_p}(U_1) = \alpha$, then $[\mathbf{F}_p(\omega) : \mathbf{F}_p] = \alpha$. If α would be even, then by lemma 2.2 and lemma 2.6 of [7], N itself would be cyclic of odd prime order, what contradicts $|N/T(W_1)| = 2$. So α must be odd, whence $\alpha|n$. Then, however, $|M|$ divides $(p^n+1)/2$, and $|M|$ divides $p^\alpha - 1$, whence also $p^n - 1$. Therefore $|M|\,|2$, and this is a contradiction. So this case γ,2,2) does not occur at all.

δ) Let $N^{\mathfrak{N}}$ be a non-trivial cyclic group. Then there is a cyclic self-centralizing subgroup X of N which is normal in N, containing $N^{\mathfrak{N}}$. We split up: δ,1) $V|_X$ is homogeneous; δ,2) $V|_X = W_1 \dotplus W_2$, the sum of two totally isotropic dual spaces.

Re δ,1) Let $V|_X$ be homogeneous. By point (1.2) of [1], V is $\mathbf{F}_p$-isomorphic to the additive group of the field K, where $K = \mathbf{F}_p(\omega)$, ω some primitive $|X|^{\text{th}}$-root of unity modulo p. By Th. II. 3.10 of [6] V is now an irreducible $\mathbf{F}_pX$-module. Then, however, the inverse image Y of X in $G/F(G)$ would act irreducibly on $F/Z(F)$, contrary to the monomiality of $YF(G)$, as proper subgroup of G. This case does not occur.

Re δ,2) By point (3.12) of [1], W_1 and W_2 are both irreducible $\mathbf{F}_pX$-modules, by and argument as done above. We have the following series:

$$N \underset{2}{\supset} T(W_1) \supseteq X \supset \{1\},$$

where $T(W_1)$ is the stabilizer of W_1 in N. Hence there exists a 2-element α such that V is an irreducible $\mathbf{F}_p(X\langle\alpha\rangle)$-module. Therefore the inverse image Z of $X\langle\alpha\rangle$ in $G/F(G)$ is equal to $G/F(G)$ (otherwise $F(G)Z$ is monomial and since $F/Z(F)$ is irreducible under the action of $X\langle\alpha\rangle$, this produces a contradiction).

Hence N/X is a cyclic 2-group. As N is not nilpotent, X contains a non-trivial cyclic Hall $2'$-subgroup N_1 such that $N_1 \lhd N$. By point (1.1) of [1], we have $X = C_N(N_1)$ or $|C_N(N_1)/X| = 2$. On the other hand, consider $\Omega_1(S_t)$, where S_t is the Sylow t-subgroup of N_1, t odd prime. Call $S = \Omega_1(S_t)$. Since $S \lhd N$, there are at most two classes of isomorphic $\mathbf{F}_pS$-modules, that is, there are at most two homogeneous constituents of S on V. Now any element of N fixes or interchanges these constituents and so $N = C_N(S)$

or $|N/C_N(S)|=2$. Hence $C_N(S)\supseteq C_N(N_1)$. Since $N/C_N(N_1)$ is a cyclic 2-group, it follows that $T(W_1)$ centralizes any such group S for any p dividing $|N_1|$. By Th. 5.3.10 of [4], $T(W_1)$ centralizes S_t, for any t dividing $|N_1|$, whence $T(W_1)$ is contained in $C_N(N_1)$. Therefore $T(W_1)=X$ or $|T(W_1)/X|=2$, whence respectively $T(W_1)=C_N(N_1)=X$ or $T(W_1)/X==C_N(N_1)/X$ has order 2. Next we split up according to the prime p which divides $|F|$.

Re δ,2,1) Let $|F|=p^{2m+1}$ with p prime and $p\equiv 3 \pmod 4$. Then $(|N|,|F|)=1$, following 1.7 (a) of [7]. Hence by lemma 2.2 and lemma 2.3 of [7], N does not contain a cyclic subgroup of order 4. Therefore any 2-element of N has order 2, whence N/X is cyclic of order 2, and N/N_1 is elementary abelian of order 4 or $X=N_1$. Let T_1 be an irreducible $\mathbf{F}_pN_1$-submodule of W_1. Since $W_1|_{N_1}$ is homogeneous, $\dim_{\mathbf{F}_p}(T_1)$ divides $\dim_{\mathbf{F}_p}(W_1)=m$. Let $\alpha=\dim_{\mathbf{F}_p}(T_1)$. Then α is the smallest natural number such that $p^\alpha\equiv 1 \pmod{|N_1|}$. See Th. II. 3.10 of [6]. As N_1 is of odd order, we see that $p^\alpha\equiv 1 \pmod{2|N_1|}$. Since $\alpha|m$, we must have in the case $|X/N_1|=2$ that $\alpha=m$ (indeed, W_1 is an irreducible $\mathbf{F}_pX$-module and m is the smallest positive integer such that $p^m\equiv 1 \pmod{|X|}$; see Th. 3.17 of [1]). Therefore there is an element $\beta\in N$ with $\beta^2=1$ such that V is an irreducible $\mathbf{F}_p(N_1\langle\beta\rangle)$-module. G is a minimal non-M-group, and so $N=N_1\langle\beta\rangle$ and $N_1=X$. According to [1], point (3.16), we can identify β with a matrix $\begin{pmatrix}0 & -\sigma\nu^{-1}\\ \sigma\nu & 0\end{pmatrix}$ with $\sigma\nu\in DK^*$, where DK^* is the semi-direct product of the group $D=\mathrm{Gal}\,(K/\mathbf{F}_p)$ with the multiplicative group K^* of K as normal subgroup. The action of D on K^* is the natural one. Further K^+ is $\mathbf{F}_p$-isomorphic to W_1, and $K=\mathbf{F}_p(\omega)$ with ω some primitive $|X|^{\text{th}}$-root of unity. Since $\beta^2=1$, we apply point (3.17) of [1], to see that β^2 corresponds to the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, where 1 as matrix entry is the unit element of K^*. On the other hand we have

$$\begin{pmatrix}0 & -\sigma\nu^{-1}\\ \sigma\nu & 0\end{pmatrix}^2=\begin{pmatrix}-\sigma\nu^{-1}.\,\sigma\nu & 0\\ 0 & -\sigma\nu.\,\sigma\nu^{-1}\end{pmatrix}=\begin{pmatrix}-\sigma^2(\nu^\sigma)^{-1}\nu & 0\\ 0 & -\sigma^2\nu^\sigma\nu^{-1}\end{pmatrix}.$$

Hence σ^2 is the trivial automorphism of K and $\nu^\sigma\nu^{-1}=-1$. Now m must be odd, as we can conclude from lemma 2.6 of [7]. Since $\mathrm{Gal}\,(K/\mathbf{F}_p)=m$, it follows that σ is the trivial automorphism of K. Then, however, $-1=\nu^\sigma\nu^{-1}=\nu\nu^{-1}=1$ contradicting char $K=p\not\equiv 0 \pmod 2$. The case δ,2,1) does not occur.

Re δ,2,2) Let $|F|=p^{2m+1}$ with p prime and $p\equiv 1 \pmod 4$. Then $(|N|,|F|)=1$. As found earlier, N/X is cyclic of order 2 or 4. So there exists some 2-element β such that $N=X\langle\beta\rangle$. As in Re δ,2,1) we may identify β with $\begin{pmatrix}0 & -\sigma\nu^{-1}\\ \sigma\nu & 0\end{pmatrix}$ where σ and ν have the same meaning as before.*) As $\beta^2\in X$, we have $\sigma^2=1$ by point (3.17) of [1]. If $\sigma=1$ then β

has order 4. If $\sigma \neq 1$, then σ maps ω to ω^{p^n}, with $\omega \in K^*$, K^* the multiplicative group of the field K, K^+ is $\mathbf{F}_p$-isomorphic to W_1, $m=2n$ with m the smallest positive integer such that $p^m \equiv 1 \pmod{|X|}$. As before, $\mathbf{F}_p(\omega)=K$ and ω is a primitive $|X|^{\text{th}}$-root of unity. Since $\beta^2 \in X$, β^2 corresponds to the matrix

$$\begin{pmatrix} -\sigma^2(\nu^\sigma)^{-1}\nu & 0 \\ 0 & -\sigma^2\nu^\sigma\nu^{-1} \end{pmatrix} = \begin{pmatrix} -(\nu^\sigma)^{-1}\nu & 0 \\ 0 & -\nu^\sigma\nu^{-1} \end{pmatrix}.$$

The order of $-\nu^\sigma\nu^{-1}$ is a power of 2, say 2^α. Let $\alpha \geqslant 1$. Now $-\nu^\sigma\nu^{-1} = -\nu^{p^n-1}$, and $\nu \neq 1$. Hence $(-\nu^{p^n-1})^{2^\alpha}=1$, so $(p^n-1)2^\alpha$ is a multiple of $|\nu|$, the order of ν. As $|\nu|=(p^{2n}-1)/\delta$ for some $\delta \in \mathbf{N}$, we have $(p^n-1)2^\alpha = s(p^{2n}-1)/\delta$, for some $s \in \mathbf{N}$. Hence $2^\alpha\delta = s(p^n+1)$. Since $p \equiv 1 \pmod 4$, we have p^n+1 divisible by 2 but not by 4. Therefore $2^{\alpha-1}|s$, and $s=2^{\alpha-1}u$, $u \in \mathbf{N}$. Then $(-\nu^{p^n-1})^2 = \nu^{(p^n-1)2} = \nu^{u(p^{2n}-1)/\delta} = \nu^{u|\nu|} = 1$. Hence, in fact, if $\alpha \geqslant 1$, then $\alpha = 1$. Therefore it follows that $\beta^2=1$ or that β is of order 4. Now, if all $\beta \in N/N_1 - X/N_1$ are of order 4, then N contains a subgroup L such that $L=\langle u, \beta | \beta^4 = u^t = 1, u^\beta = u^{-1}, t \text{ odd prime}, t \neq p\rangle$. By lemma 2.2 and lemma 2.6 of [7] we have first the necessary condition that p has odd order modulo t, and then we see that we have here $N=L$, thereby settling the case 4) of theorem 2.

So we may assume that any element of $N/N_1 - X/N_1$ has order 2. Then assume $N = X\langle\beta\rangle$ with $\beta^2=1$. Now X/N_1 is a non-trivial cyclic 2-group. Indeed, if $X=N_1$ then we would have the necessary condition that m is odd, see lemma 2.6 of [7] and on the other hand, since $\beta^2=1$, $\sigma \neq 1$ which contradicts the oddness of m, as $m=2n$, see above. Therefore N/N_1 is not an abelian 2-group, but it must be a dihedral 2-group, where it can also happen that N/N_1 is a Klein four group. By abuse of language we say: N/N_1 is a dihedral 2-group. Next we study the structure of N more closely. By [1], we may identify $\varrho \in X$ such that $\langle\varrho\rangle = X$ with the matrix $\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1}\end{pmatrix}$, where $K=\mathbf{F}_p(\omega)$ is the field of p^m elements. Moreover $p^m \equiv 1 \pmod{|X|}$. By what we have said before, $m=2n$. Calculate (with $\beta^2=1$):

$$\begin{pmatrix} 0 & -\sigma\nu^{-1} \\ \sigma\nu & 0 \end{pmatrix}\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}\begin{pmatrix} 0 & -\sigma\nu^{-1} \\ \sigma\nu & 0 \end{pmatrix} = \begin{pmatrix} (\omega^{-1})^\sigma & 0 \\ 0 & \omega^\sigma \end{pmatrix} = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}^{-p^n}.$$

Hence $N = \langle x, u | u^2 = 1 = x^s,\ p^{2n} \equiv 1 \pmod s,\ p \equiv 1 \pmod 4,\ x^u = x^{-p^n}\rangle$. Let $s = 2^\gamma s'$ with $(2^\gamma, s')=1$. By the dihedral structure of the Sylow 2-subgroup of N we have: $x^{s'} \overset{u}{\longmapsto} x^{-s'} = x^{-s'p^n}$. Hence $x^{-s'(p^n-1)}=1$ and therefore $s'(p^n-1)$ is a multiple of s, the order of x. So $2^\gamma | p^n - 1$. Let n_1 be the (odd!) order of N_1. Then, by lemma 2.2 and lemma 2.3 of [7], the smallest positive integer α such that $p^\alpha \equiv 1 \pmod{n_1}$ is odd. As $W_1|_{N_1}$ is homogeneous, α divides $m=2n$. Hence $\alpha | n$. Therefore $p^n \equiv 1 \pmod{n_1}$, $p^n \equiv 1 \pmod{2^\gamma}$, whence also $p^n \equiv 1 \pmod{2^\gamma n_1}$, that is $p^n \equiv 1 \pmod{|X|}$.

This contradicts the irreducible action of $T(W_1)=X$ on the $\mathbf{F}_pX$-module W_1 with $\dim_{\mathbf{F}_p}(W_1)=2n=m$. See theorem II. 3.10 of [6]. The case Re δ,2,2) has been completed now.

Re δ,2,3) Let $|F|=2^{2m+1}$. If N has odd order then N is of odd prime order and nilpotent. So N must be of even order. By Th. 1.7 of [7], N/N_1 is a cyclic 2-group. Here we have $N=T(W_1)\langle\beta\rangle$, with β corresponding to the matrix $\begin{pmatrix}0 & \sigma\nu^{-1}\\ \sigma\nu & 0\end{pmatrix}$, the entries have the same meaning as above, with p replaced by 2. Then $\sigma^2\nu^{-\sigma}\nu\in K^*$. Hence $\sigma=1$ or $\sigma\colon\varepsilon\mapsto\varepsilon^{2^n}$ and $\nu^{2^n-1}\in K^*$ with $m=2n$. Assume that $\sigma\neq 1$. Then we find that

$$N=\langle x,\,u\,|\,u^{2^l}=1=x^s,\ 2^{2n}\equiv 1 \pmod{s},\ x^u=x^{-2^n}\rangle.$$

Let $s=|x|=2^\delta s'$ with $(2,s')=1$. Then $x^{s'}\xmapsto{u} x^{s'}=x^{-2^n s'}$. It follows that $x^{s'(2^n+1)}=1$. Therefore $s'(2^n+1)$ is a multiple of s. Hence $\delta=0$ and $l=1$. Therefore $T(W_1)=N_1$. According to Th. II. 3.10 of [6], $2^m\equiv 1 \pmod{|N_1|}$ with m odd. So not only the case $\sigma\neq 1$ does not occur, but also, as $F(G/F(G))$ is a $2'$-group by the beginning of this paper, we must have $\sigma=1$ and $\beta^2=1$, N/N_1 cyclic of order 2. For any $\omega\in K^*$ such that $K=\mathbf{F}_2(\omega)$, $|\omega|=$order of N_1, we obtain that N is a dihedral group; this follows from $\begin{pmatrix}0 & \nu^{-1}\\ \nu & 0\end{pmatrix}\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}\begin{pmatrix}0 & \nu^{-1}\\ \nu & 0\end{pmatrix}=\begin{pmatrix}\omega^{-1} & 0\\ 0 & \omega\end{pmatrix}$. See (3.17) of [1]. N_1 is inverted by β. Now we apply the following procedure, derived from Berger's paper [1]. We can view V as a 2-dimensional $\mathbf{F}_{2^m}N$-module V_1 with basis $\{e_1, e_2\}$ equipped with a symplectic form $\bar g\left(\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)$, which is left invariant under the action of N, and such that $\bar g\left(\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)= =(\beta_1\gamma_2-\beta_2\gamma_1)$. Moreover $\mathrm{Tr}_{\mathbf{F}_{2^m}\to\mathbf{F}_2}(\bar g)$ is just the symplectic form described in § 1. This form is also invariant under the action of N, where $N=\left\langle\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}, \begin{pmatrix}0 & \nu^{-1}\\ \nu & 0\end{pmatrix}\right\rangle$. Next we argue that we may take $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ instead of $\begin{pmatrix}0 & \nu^{-1}\\ \nu & 0\end{pmatrix}$. Since the characteristic of K is 2, there is $\delta\in K^*$ such that $\delta^{-2}=\nu$. Look at the basis $\{\delta\nu e_1, \delta e_2\}$, obtained by applying the $\mathbf{F}_{2^m}$-linear map $\psi\colon V_1\to V_1$, defined by $\psi(e_1)=\delta\nu e_1$, $\psi(e_2)=\delta e_2$. Let $e_1'=\delta\nu e_1$ and $e_2'=\delta e_2$. Then, with respect to that basis, β corresponds to $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and x corresponds again to $\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}$. With respect to the basis $\{e_1', e_2'\}$ the symplectic form $\bar g$ remains invariant under ψ, as can be seen from

$$\bar g(\beta_1e_1+\beta_2e_2,\gamma_1e_1+\gamma_2e_2)=\beta_1\gamma_2-\gamma_1\beta_2=\delta^2\nu(\beta_1\gamma_2-\gamma_1\beta_2)=\beta_1\delta\nu\delta\gamma_2-\beta_2\delta\gamma_1\delta\nu=$$
$$=\bar g(\beta_1\delta\nu e_1+\beta_2\delta e_2,\ \gamma_1\delta\nu e_1+\delta\gamma_2e_2)=\bar g(\psi(\beta_1e_1+\beta_2e_2),\ \psi(\gamma_1e_1+\gamma_2e_2)).$$

Therefore we can assume that $\psi_0(\beta)=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, and that x corresponds to $\psi_0(x)=\begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}$, where ψ_0 is the map as described in point (3.17) of [1]. So indeed we may assume that β corresponds to $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. Now, let $|\omega|=tp$, p odd prime. Assume that α is the smallest positive integer such that $2^{\alpha}\equiv 1 \pmod{p}$. By homogeneouity, α divides m. Let $\eta=\omega^t$. Let ϱ be a generator of $\mathbf{F}_{2^\alpha}{}^*$. Then $R=\left\{\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \,\middle|\, \eta_i \in \langle\varrho\rangle \cup \{0\}\right\}$ spans a 2-dimensional vector space T over $\mathbf{F}_{2^\alpha}$, left invariant by the elements of the group $N_2=\langle \begin{pmatrix}\eta & 0\\ 0 & \eta^{-1}\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\rangle$, where we can assume without loss of generality as we saw above that the elements of R are taken with respect to the basis $\{e_1, e_2\}$, and also the corresponding elements of the "matrix" group N_2 are considered as elements of N. Now $\bar{g}\left(\begin{pmatrix}\eta\\ 0\end{pmatrix}, \begin{pmatrix}0\\ \eta^{-1}\end{pmatrix}\right)=1$, and $\mathrm{Tr}_{\mathbf{F}_{2^m}\to \mathbf{F}_2}(1)=$ $=\sum_{i=0}^{m-1} 1^{\lambda^i}=m$ where $\mathrm{Gal}\,(\mathbf{F}_{2^m}/\mathbf{F}_2)=\langle\lambda\rangle$. Since m is odd, we find m unequal to 0, viewed as element of $\mathbf{F}_2$. Now, let Ψ_0 as in [1], be the monomorphism of the $\mathbf{F}_2$-module V on the $\mathbf{F}_{2^m}$-module V_1 (It is a $\mathbf{F}_2$-isomorphism). By [1] we have $\psi_0(x)\Psi_0(u)=\Psi_0(xu)$, all $x\in N$, $u\in V$. Consider $R_1=\Psi_0{}^{-1}(R)$. Then N_2 acts on R_1 as $\mathbf{F}_2N_2$-module. The "definition" of this action is $xu=\Psi_0{}^{-1}(\psi_0(x)\Psi_0(u))$. Therefore R_1 is an irreducible non-singular $\mathbf{F}_2N_2$-submodule of V. By minimality, $N=N_2$ and $R_1=V$. The case 3) of theorem 2 has been settled and the proof of the whole theorem is complete.

§ 3. THE ACTION OF $G/F(G)$ ON F

In this section we determine all the minimal solvable non-M-groups G such that $G/F(G)$ satisfies the hypotheses of theorem 2. Most conclusions drawn are immediately extracted from § 2.

1) Let $G/F(G)$ be cyclic of prime order q. Then q is odd. In this case we will give an existence proof for G. Here we have $F=F(G)$ with F an extra-special p-group of exponent p if p is an odd prime, or F is an extra-special 2-group but not dihedral, if $p=2$. Let $|F|=p^{2m+1}$. Then p has order $2m$ modulo q and $p^m\equiv -1 \pmod{q}$. By point (2.1) of [1] we can construct G as follows: $F/Z(F)$ is $\mathbf{F}_p$-isomorphic to the additive group of the field K with p^{2m} elements. Let $\varphi\colon \varepsilon \mapsto \varepsilon^{p^m}$, $\varepsilon\in K$, be the automorphism of order 2 of K, and let T be the multiplicative group in K^* of order p^m+1. Now if G exists, then $G/F(G)$ can be considered as a subgroup of T, where the action of $G/F(G)$ on F is compatible with the action of the subgroup S of T of order q, namely compatible with the action of T on K by multiplication. See here Th. II. 3.10 of [6] and point (2.1) of [1]. So what we must do is the following: We have to find the

action of S on K (with régard to the invariance of some symplectic form), and then extend this action to our original F. The relative holomorph $F\langle S\rangle$ is then the required group G as is easy to see. As said above, T acts by multiplication upon K. Choose $\mu \in K^*$ with $\mu^\varphi = -\mu$ (if $p=2$ let $\mu = 1$). Put $g(u, v) = \mathrm{Tr}_{\mathbf{F}_{p^m} \to \mathbf{F}_p}(\mu(uv^\varphi - u^\varphi v))$, $u, v \in K$. Then $g(u, v)$ is a symplectic form on K, fixed by T. Hence T is a subgroup of $Sp(2m, p)$. Take our q as divisor of $p^m + 1$ and consider the subgroup S of T of order q, as suggested above. Then S is an irreducible subgroup of $Sp(2m, p)$.

Now we have to prove that the action of S on K can be extended to an action of F. First of all, there is the action of $G/F(G)$ now, on the factor group $F/Z(F)$. See again point (2.1) of [1]. This action leaves the symplectic form on $F/Z(F)$ invariant, and this symplectic form corresponds to the symplectic form defined above. Now, if p is odd, then the action of S on $F/Z(F)$ can be extended to the whole of F, by a theorem of Dickson, viz. [9]. If $p=2$, then Aut (F)/Inn (F) is isomorphic to the full orthogonal group $0_\varepsilon(2m, 2)$, leaving the appropriate quadratic form on $F/Z(F)$ invariant; see [3]. We have

$$|0_\varepsilon(2m, 2)| = 2^{m^2-m+1}(2^m - \varepsilon) \prod_{i=1}^{m-1} (2^{2i} - 1),$$

and $0_\varepsilon(2m, 2)$ is a subgroup of $Sp(2m, 2)$, where

$$|Sp(2m, 2)| = 2^{m^2} \prod_{i=1}^{m} (2^{2i} - 1);$$

see [9]. Now 2 has here order $2m$ modulo q and $\varepsilon = +1$, if F is the amalgamated product of m dihedral groups of order 8, and $\varepsilon = -1$, if F is the amalgamated product of one quaternion group of order 8 with $m-1$ dihedral groups of order 8. Therefore, by the order formula for $0_\varepsilon(2m, 2)$, we must have the latter F for which $\varepsilon = -1$. Notice that any Sylow q-subgroup of $0_{-1}(2m, 2)$ is cyclic and equal to any Sylow q-subgroup of $Sp(2m, 2)$, i.e. up to isomorphism. Hence there exists a subgroup S_1 of Aut (F) of order q, whose action on $F/Z(F)$ coincides with the action of S on K^+.

Hence S_1 acts on F and the existence of G has been established in all cases for p. That G is indeed a minimal non-M-group, is left to the reader. The order of G equals qp^{2m+1}.

2) Let $G/F(G)$ be quaternion of order 8. Then $F = F(G)$ and F is an extra-special p-group of exponent p, such that p is an odd prime with $p \equiv 1 \pmod 4$. Hence $|F| = p^3$. The example can be derived from point (2.3) of [1]. Up to isomorphy there is only one group, namely:

$$\begin{aligned} G = F\langle \alpha, \beta\rangle = \langle x, y, \alpha, \beta \mid\, & x^p = y^p = z^p = \alpha^4 = \beta^4 = 1,\ [x, y] = z, \\ & [x, z] = [y, z] = 1,\ x^\alpha = x^t,\ y^\alpha = y^{t^{-1}},\ t^2 \equiv -1 \pmod p, \\ & x^\beta = y,\ y^\beta = x^{-1},\ \beta^\alpha = \beta^{-1},\ \alpha^2 = \beta^2,\ p \equiv 1 \pmod 4\rangle. \end{aligned}$$

3) Let $G/F(G)$ be cyclic of order 4. Then $F=F(G)$ and F is an extra-special p-group of exponent p, such that p is an odd prime with $p\equiv -1$ (mod 4). Hence F has order p^3. Let $F=\langle x, y | x^p=y^p=z^p=1,\ [x, y]=z, [x, z]=[y, z]=1\rangle$. We may take $G=F\langle c\rangle$, with $c^4=1$, $x^c=y$, $y^c=x^{-1}$. It is clear that G is a minimal non-M-group.

4) Let $G/F(G)=\langle u, \beta | \beta^4=u^t=1, t \text{ odd prime}, u^\beta=u^{-1}\rangle$. Then $F=F(G)$ and F is an extra-special p-group of exponent p with $p\neq t$, p odd prime. Moreover $p\equiv 1$ (mod 4). Let $|F|=p^{2m+1}$. Then l, the order of p mod t, is odd as we saw earlier. But then $p^l\equiv 1$ (mod $2t$), i.e. $l=m$. We give an existence proof for G. Consider $\langle u, \beta\rangle$. Then

$$\langle u, \beta\rangle \cong \left\langle \begin{pmatrix}\omega & 0\\ 0 & \omega^{-1}\end{pmatrix}, \begin{pmatrix}0 & -\nu^{-1}\\ \nu & 0\end{pmatrix} \middle|\ \text{fixed } \nu\in \mathbf{F}_{p^m},\ \langle\omega\rangle\cong\langle\beta^2, u\rangle,\ \omega\in\mathbf{F}_{p^m}\right\rangle.$$

Call the latter matrix group $\bar{G}$. The symplectic form $\bar{g}\left(\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}, \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix}\right)= =\beta_2\gamma_1-\beta_1\gamma_2$ is fixed by $\bar{G}$. Hence $\bar{G}$ is a subgroup of $Sp(2, p^m)$, whence also a subgroup of $Sp(2m, p)$ by Th. II. 9.24 of [6]. Then by [9], $\bar{G}$ can be considered as subgroup of Aut (F). Build the semi-direct product $\bar{G}F$. This is the required group G. It is easy to see that G is a minimal non-M-group. We omit the trivial verification.

5) Assume that $G/F(G)$ is dihedral of order $2q$, q odd prime. Then $|F|=2^{2m+1}$, where the order of 2 modulo q equals m. Here we will meet the complication that $F(G)\neq F$, in general. By [8], $F(G)$ is the central product of a cyclic 2-group C with F, such that $\Omega_1(C)$ is identified with $Z(F)$. In fact, $C=Z(G)$. As m is odd, see above in Re δ,2,3), it follows by an inspection of the order formula for the group $0_\varepsilon(2m, 2)$ that we must have $\varepsilon=+1$, that is, F must be the central product of m dihedral groups of order 8. Now, if we can prove that $\langle a, b | a^q=1=b^2, bab^{-1}=a^{-1}\rangle$ can be represented faithfully in $0_1(2m, 2)$ then we know (by [9]) that $\langle a, b\rangle$ can be seen as subgroup of Aut (F). When it has been established, then we have to give the generators and relations for G, that is to give a practical device for constructing such a G. As we saw in Re δ,2,3), $\langle a, b\rangle$ can be faithfully represented as subgroup of $Sp(2, 2^m)$, whence also as subgroup of $Sp(2m, 2)$. Let $F/Z(F)=\langle \bar{x}_1, \dots, \bar{x}_m, \bar{y}_1, \dots, \bar{y}_m\rangle$, where the bar denotes class modulo $Z(F)$. Assume that this basis is given so that $[x_i, y_j]=z^{\delta_{ij}}$, δ_{ij} Kronecker delta, $[x_s, x_t]=[y_s, y_t]=1$ for all $s, t\in\{1, 2, \dots, m\}$. Although we don't need it, we stipulate that all the x_i and y_j written, are of order 2. The representation of $\langle a, b\rangle$ as subgroup of $Sp(2m, 2)$ with respect to the given basis, is now as follows:

$$a \mapsto \begin{pmatrix}A & 0\\ 0 & B\end{pmatrix},\quad b\mapsto \begin{pmatrix}0 & D\\ C & 0\end{pmatrix},\quad A, B, C, D\in GL(m, 2).$$

As $\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\in Sp(2m, 2)$, it follows that $B=(A^*)^{-1}$, where A^* is the transpose of A.

Further, as

$$[x_i, y_j] = \prod_{t=1}^{m} ([y_t, x_t]^{\varepsilon_{ti}\varphi_{tj}}),$$

for $(\varepsilon_{ij}) = C$, $(\varphi_{ij}) = D$, we see that $\sum_{t=1}^{m} \varepsilon_{ti}\varphi_{tj} = \delta_{ij}$, the Kronecker delta. So $D^* = C^{-1}$. On the other hand, $\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}$ is a matrix of order 2, so $D = C^{-1}$. Thus C is symmetric. Also we see without using it, that $C^{-1}A^*C = A$. Next we show that $\begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & C^{-1} \\ C & 0 \end{pmatrix}$ are both contained in $0_1(2m, 2)$. Therefore we have to show that both matrices leave invariant the quadratic form defined by

$$q(\bar{x}_1^{\xi_1}\bar{x}_2^{\xi_2} \dots \bar{x}_m^{\xi_m}\bar{y}_1^{\eta_1} \dots \bar{y}_m^{\eta_m}) = \sum_{=1}^{m} \xi_i\eta_i, \quad \xi_s, \eta_t \in \mathbf{F}_2.$$

Let $(\alpha_{ij}) = A$, $(\beta_{ij}) = B$, $(\varepsilon_{ij}) = C$, $(\varphi_{ij}) = D$. Then we have:

$$\sum_{i=1}^{m} \xi_i\eta_i \mapsto \sum_{i=1}^{m} \Big(\sum_{t=1}^{m} \alpha_{it}\xi_t\Big)\Big(\sum_{s=1}^{m} \beta_{is}\eta_s\Big) =$$

$$= \sum_{\substack{t=1\\s=1}}^{m} \Big(\sum_{i=1}^{m} \alpha_{it}\beta_{is}\Big)\xi_t\eta_s = \sum_{\substack{t=1\\s=1}}^{m} \delta_{ts}\xi_t\eta_s = \sum_{j=1}^{m} \xi_j\eta_j,$$

as $B = (A^*)^{-1}$.
And, as $C = C^*$, $D = C^{-1}$:

$$\sum_{i=1}^{m} \xi_i\eta_i \mapsto \sum_{i=1}^{m} \Big(\sum_{t=1}^{m} \varphi_{it}\eta_t\Big)\Big(\sum_{s=1}^{m} \varepsilon_{is}\xi_s\Big) = \sum_{\substack{t=1\\s=1}}^{m} \Big(\sum_{i=1}^{m} \varphi_{it}\varepsilon_{is}\Big)\eta_t\xi_s =$$

$$= \sum_{\substack{t=1\\s=1}}^{m} \Big(\sum_{i=1}^{m} \varepsilon_{si}\varphi_{it}\Big)\eta_t\xi_s = \sum_{\substack{t=1\\s=1}}^{m} \delta_{ts}\eta_t\xi_s = \sum_{j=1}^{m} \eta_j\xi_j.$$

Hence $\langle a, b\rangle$ can be faithfully represented in $0_1(2m, 2)$. By the construction of [9] (and using that the chosen x_i, y_j are all of order 2 (here we need it!)), we see that $\langle a, b\rangle$ can be viewed as subgroup of Aut (F). Notice that the argument is here more delicate than in the odd prime case, caused by the fact that in the $p=2$ case Aut (F) does not split over Inn (F) in general, where Aut (F)/Inn $(F) \simeq 0_\varepsilon(2m, 2)$. See [5].

Now we come to the description of the group G. The following normal series occurs:

$$G \overset{2}{\supset} R \overset{q}{\supset} F(G) = C \curlyvee F \overset{2^s}{\supseteq} F \supset Z(F) \supset \{1\}.$$

In order that G be a minimal non-M-group, we know that G/F has cyclic Sylow 2-subgroups. So there exists an element $t \in G$, with $t^{2^{s+1}} \in F$, $t^{2^s} \notin F$, where 2^{s+1} is the order of C and $s \geqslant 0$. Now $t^2 = cf$, for some $c \in C$ and $f \in F$. Assume that $s \geqslant 1$. Then $t^4 \in C$. So $t^{2^{s+1}}$ is contained in $F \cap C = Z(F)$.

If $s \geqslant 1$, and t has order 2^{s+1}, then $F(G)$ would be of exponent 2^{s+1}. By construction t^2 acts trivially on F, so t^2 centralizes $\langle F, t^2 \rangle = F(G)$. So $t^2 \in C$. The order of t^2 would be at most 2^s contrary to the order of C. So, if $s \geqslant 1$, then $t^{2^s+1} = z$, with $\langle z \rangle = Z(F)$, $C = \langle t^2 \rangle$. t induces an automorphism of order 2 on F, as constructed earlier, and the construction of R is also clear. If $s=0$, then $F(G)=F$ and t^2 centralizes F, so $t^2=1$ or $t^2=z$. The last remark settles the structure of G completely, namely, R cannot be anything else as the semi-direct product of $F(G)$ with a cyclic group of order q (by the Schur-Zassenhaus theorem), and the action of $R/F(G)$ on F is constructed above, and t acts on F corresponding to the action of the matrix $\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}$.

REMARK. In a future paper we hope to get rid of the additional assumption on $G/F(G)$, made in the hypotheses of theorem 2.

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*) See now point (1.1) of [1].